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Journal of Applied Mathematics and Mechanics 70 (2006) 945–955
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Equilibrium cracks in composites reinforced with unidirectional fibres☆

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Received 1 March 2005

Abstract

A discrete-continuum approach, proposed by Novozhilov for analysing the equilibrium states of a brittle of crack in an isotropic body, is applied to a penny-shaped crack situated in a fibre-reinforced composite perpendicular to the fibres. The structural non-uniformity of the material is taken into account by the presence of unbroken fibres in the narrow part of the crack, adjoining the edge, and the different effect of the strength properties of the fibre and matrix on the limit state of the crack. Using this model, the range of dimensions of equilibrium cracks is established and an estimate is given of the critical size of the bridged part of the crack, corresponding to the onset of catastrophic fracture. It is shown that this dimension has the same value for a penny-shaped crack and for a crack under plane strain, does not depend on the form of the load and, under the condition of its smallness, is a brittle fracture characteristic of a fibre-reinforced material. The possibility of using this fracture model for two types of ceramics is analysed on the basis of experimental data.

One of the methods of increasing the fracture toughness of brittle materials, such as ceramics, is to reinforce them with fibres, which connect the surfaces of the crack and hence, slow down its prapagation.[1] A considerable number of papers (see, for example, Refs 1–15) have been devoted to investigating the behaviour of a tensile crack, situated in a fibre-reinforced composite material perpendicular to the fibres. Experiments show that such a crack propagates in its own plane when parts of unbroken fibres are present (the bridging zone), adjacent to the crack edges.[3] The size of the bridging zone of a growing crack depends on the mechanical properties of the fibre and the matrix, and also on the adhesion strength of their connection.[2–4]

The most important problem, the solution of which precedes an analysis of the behaviour of a crack in fibre-reinforced material, is to establish the law of deformation of the fibre in the bridging zone, taking into account its pull-out from the matrix. As research shows, the law of the fibre pull-out from the matrix for such brittle materials as a ceramics, has a strengthening form.[5–8] However, taking into account the fact that the limit state of the crack in the ceramics corresponds to a small zone of unbroken fibres, this law for fibres in the bridging zone can be taken, with a sufficient degree of accuracy, in the form of a step function. Moreover, the smallness of the bridging zone enables the unknown forces of interaction of the surfaces of the crack to be replaced by constant forces (see, for example, Ref. 4 and the references given there).

☆ *Prikl. Mat. Mekh*. Vol. 70, No. 6, pp. 1054–1066, 2006
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doi:10.1016/j.jappmathmech.2007.01.011

In this formulation, the model of a crack in a composite with a small zone of unbroken fibres is related to well-known models,[16,17] but differs from them by the stress singularity at the edge of the crack. A combination of the two criteria (the condition for the fibre to break and the condition for the crack to propagate within the framework of linear fracture mechanics) enables us, in this model, to give an estimate of the size of the bridging zone and the dimensions of the crack in the limit state.[5,11]

The possible range of diameters of equilibrium penny-shaped cracks in a uniform isotropic material was established in Ref. 18 from the point of view of Novozhilov's discrete-continuum theory.[19,20] Novozhilov's investigation correlates with the well-known Thomson theory on lattice trapping in a range of external forces which ensure the equilibrium state of the crack,[21–23] and enables progress to be made in solving a number of applied problems.[13,24]

In the Novozhilov-Thomson approach the interatomic-interaction forces at the edge of a microcrack are taken into account. To extend this approach to a macrocrack in a uniform body necessitates a consideration of the cohesion forces, the nature of which does not have a sufficiently clear physical meaning, may have a different interpretation and was therefore not previously specified.[18] However, we will apply the Novozhilov-Thomson approach in the best way possible to ceramics-type composites with a brittle matrix, reinforced with unidirectional fibres. The cohesion forces in this cases will be the natural resistance forces of the fibres, connecting the surfaces of the so-called bridged crack and its edge.[13]

Unlike our previous paper,[18] below we consider a penny-shaped crack in a composite, reinforced with parallel fibres and, on average, having the properties of an elastic transversely isotropic body. The non-uniformity of the material, i.e. the discreteness of its structure, is taken into account by using a hybrid Novozhilov-Thomson method. The crack growth is determined by two factors: breaking of the outer fibres in the bridged part of the crack and the fracture of the brittle matrix in a certain zone in front of the crack. Here we assume that the deformation and strength properties of the fibre and the matrix, and also the crack size, agree with the requirement for the cohesion zone to be small. This implies that a large number of the fibres which bridge the crack must be broken, before the crack starts to propagate. This requirement narrows down the class of materials to be considered and sets a lower limit to the dimensions of equilibrium bridged cracks, but, as the examples presented below show, it is justified for cracks in a ceramics.

1. Formulation of the problem

Consider an elastic brittle-matrix composite (for example, ceramics) with a penny-shaped crack

$$\Gamma = \{(x_1, x_2, x_3) : x_3 = 0,\ \rho^2 = x_1^2 + x_2^2 \le a^2\}$$

situated in a plane perpendicular to the fibres (Fig. 1). We will assume that, in general, the crack is opened by infinitely distant forces $\sigma_{33}^{\infty} = p$ and two oppositely directed concentrated normal forces P, applied at the centre, and is in one of the last stages of development, when unbroken fibres remain only in a narrow ring-shaped region

$$\Gamma_0 = \{(x_1, x_2, x_3) : x_3 = 0,\ b \le \rho \le a\}$$

of width $\Delta = a - b$ (Fig. 2).

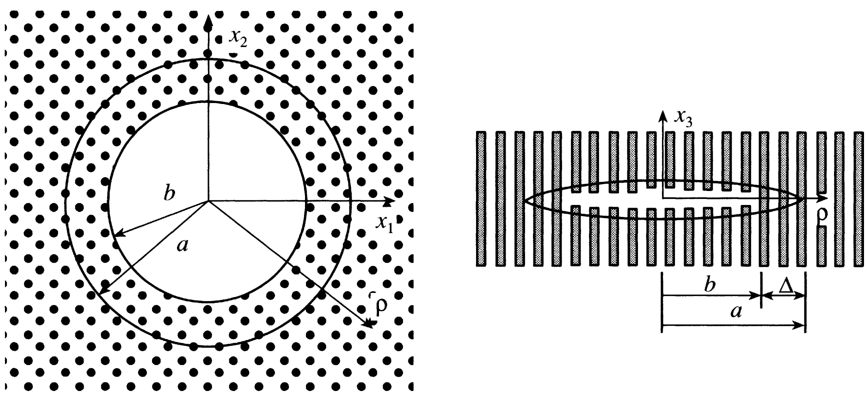

Fig. 1.

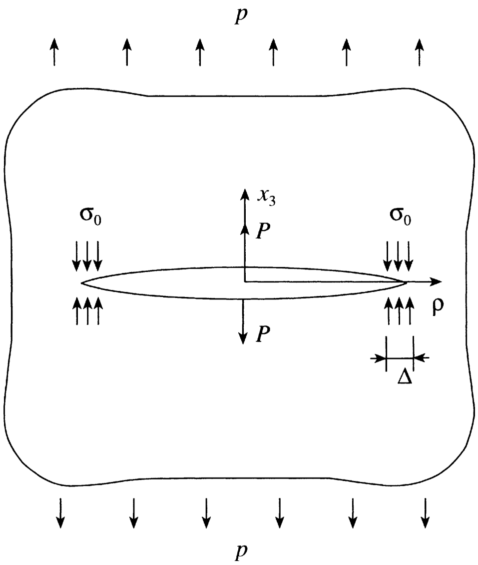

Fig. 2.

To determine the stress-strain state of the composite with the crack we will assume that the composite behaves in the same way as a certain uniform transversely isotropic material. We will determine the constants of elasticity of this material later, starting from the elastic properties of the isotropic components of the composite, i.e. the fibre and the matrix and the fibre distribution density. We will neglect slippage between the fibres and the matrix in the continuous part of the composite.

We will model the reaction of the fibres in the cohesion zone Γ_0 when the crack is opened by the action of a constant normal force σ_0 (Fig. 2). Here the outermost fibres (when $\rho = b$) are in the limit state, which is given by the equation

$$w(b) = w_0 \tag{1.1}$$

where $2w$ is the opening of the crack in the direction of the x_3 axis and w_0 is a certain constant.

The inequality $w(\rho) > w_0$ denotes that, on a circle of radius ρ, the fibres are broken. Hence, $w(\rho) > w_0$ when $\rho < b$ and $w(\rho) > w_0$ when $b < \rho \le a$.

We determine the quantity σ_0 from the deformation law of the fibre $\sigma = \sigma(w)$ when it is pulled out of the matrix (Fig. 3), namely

$$\sigma_0 = \frac{c}{w_0 - w_B}\int_{w_B}^{w_0} \sigma(t)dt \tag{1.2}$$

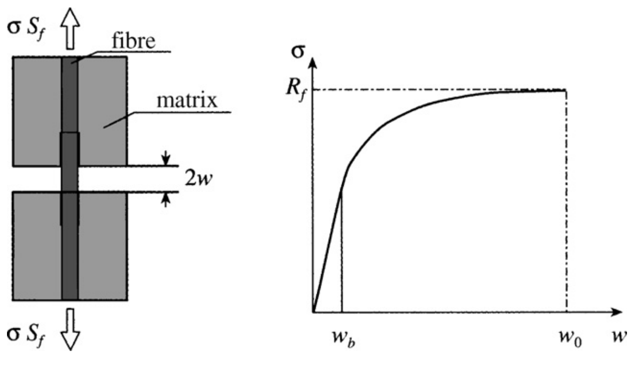

Fig. 3.

where $\sigma(w)$ is the force in the fibre, w_B is the transition point to the non-linear part of the deformation and c is the volume fraction of the fibres in the composite. The constants w_B and w_0 depend on the fibre-matrix adhesion properties, in particular, the resistance to delamination by a shear force, and also on the moduli of elasticity of each component.

The meaning of equality (1.2) is that the work of the force σ_0 when the crack opens on the non-linear part of the deformation of the outermost fibre is identical with the work done by the force σ in stretching the fibre on the same part of the deformation, taking into account the concentration of fibres in the composite. This replacement of the true relationship $\sigma(w)$ by the constant quantity σ_0 from (1.2) is similar to Novozhilov's replacement of the descending branch of the law of interaction of the atoms by a step function.[19,20] In fact, the force σ_0, which opposes the opening of the crack, changes on the segment $b \le \rho \le a$ from the maximum value R_f (the tensile strength of the fibre) when $\rho = b$ to a certain minimum value when $\rho = a$, but, as calculations show (Ref. 2, Fig. 2), over a large part of the cohesion zone this change can be neglected.

Following Novozhilov's idea, we will assume that the crack is in equilibrium if

$$\int_a^{a+D} \sigma_{33}^m(\rho)d\rho \le DR_m \tag{1.3}$$

where σ_{33}^m is the stress in the matrix, R_m is the tensile strength of the matrix and D is the size of the matrix damage zone. Violation of condition (1.3) indicates that the crack is growing. When the equality in condition (1.3) is satisfied, the crack is in a critical state.

When there is no slippage between the fibre and the matrix in the continuum part of the composite, the mean stress σ_{33} is related to the corresponding stresses in the fibre and the matrix (σ_{33}^f and σ_{33}^m) by the equations[2]

$$\sigma_{33}/E_3 = \sigma_{33}^f/E_f = \sigma_{33}^m/E_m; \; E_3 = cE_f + (1-c)E_m \tag{1.4}$$

where E_f and E_m are Young's moduli of the fibre and the matrix respectively.

Taking relations (1.4) into account, we will write condition (1.3) for the crack to grow in the form

$$\int_a^{a+D} \sigma_{33}(\rho)d\rho \le DR; \quad R = R_m E_3/E_m, \quad D = kD_m, \quad D_m \equiv \frac{2}{\pi}\left(\frac{K_{1c}^m}{R_m}\right)^2 \tag{1.5}$$

The expression for D is written for the case of brittle fracture of the matrix. Here K_{1c}^m is the fracture toughness of the matrix material and k is a coefficient representing the effect of the nonuniformity of the stress-strain state in the neighbourhood of the fibre, and also the effect of other factors, related to fiber-matrix adhesion properties, on the fracture of the matrix.

The equality $D = D_m$ arises from the equivalence of Novozhilov's criterion for brittle fracture and Irwin's force criterion,[25] as it applies to a uniform isotropic material possessing the properties of the matrix. If the equality $D = D_m$ is also satisfied for the composite, this indicates that the presence of fibres has no effect on the fracture of the matrix. An analysis of the dependence of the main fracture characteristics of a composite with a crack on the dimensions of the fracture zone D will be presented at the end of this paper.

2. Fundamental relations

As previously in Ref. 18, basing ourselves on well-known solutions[26,27] and assuming that the crack is in the equilibrium state in a uniform transversely isotropic material acted upon by the forces indicated above (Fig. 2), we arrive at the following relations for the opening of the crack $2w$ and the stress σ_{33} in the $x_3 = 0$ plane

$$w(\rho) = 2H\left\{2p\sqrt{a^2-\rho^2} - 2\sigma_0 F(\rho) + \frac{P}{\pi\rho}\operatorname{arctg}\frac{\sqrt{a^2-\rho^2}}{\rho}\right\}, \; \rho \le a \tag{2.1}$$

$$\sigma_{33}(\rho) = \frac{2}{\pi}\left[\frac{ap-\sigma_0\sqrt{a^2-b^2}}{\sqrt{\rho^2-a^2}} + \frac{\pi p}{2} - p\arcsin\frac{a}{\rho} + \sigma_0\arcsin\sqrt{\frac{a^2-b^2}{\rho^2-b^2}}\right] + \\ + \frac{Pa}{\pi^2\rho^2\sqrt{\rho^2-a^2}}, \quad \rho > a \tag{2.2}$$

Here

$$F(\rho) = \int_r^a \sqrt{\frac{t^2-b^2}{t^2-\rho^2}}dt, \quad r = \begin{cases} b, & 0 \le \rho \le b \\ \rho, & b \le \rho \le a \end{cases}$$

The coefficient H, characterizing the stiffness of the anisotropic material, is found from the formula[27]

$$H = \frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33}-A_{13}^2)}, \quad \gamma_k^2 = \frac{m_kA_{33}}{m_kA_{44}+A_{13}+A_{44}} = \frac{A_{44}+m_k(A_{13}+A_{44})}{A_{11}}, \quad k = 1, 2 \tag{2.3}$$

where A_{jj} are the components of the elastic compliance tensor of the material along the principal axes of anisotropy. For materials reinforced with parallel cylindrical fibres the constants A_{jj} can be expressed in terms of the moduli of elasticity and the volume fractions of the components.[28]

In components based on a ceramics, Young's moduli E_f and E_m usually differ by less than a factor of three, while Poisson's ratios (υ_f and υ_m respectively) are equal. As a consequence of this the composite possesses weak anisotropy of the elastic properties and the following simpler formula than (2.3) is usually employed for it (see, for example, Refs 2,5)

$$H = \frac{1-\nu_f^2}{\pi E_3} \tag{2.4}$$

The error in calculating H using this formula for ceramic materials is comparable with the error in employing the method indicated above. According to the second formula of (1.4) when $E_f = E_m$, Eq. (2.4) is identical with the analogous relation for isotropic materials.[18] Below all the results are obtained using formula (2.4).

In order to obtain the conditions which ensure the equilibrium state of the penny-shaped crack (of diameter d) in the composite, we will substitute expression (2.1) into (1.1), and expression (2.2) into condition (1.5), assuming $\Delta/a \ll 1$ and $D/a \ll 1$. We then obtain, respectively,

$$\varsigma\left(\frac{d}{D}\right) = g_1^a(\alpha), \quad g_1^a(\alpha) = \frac{\beta}{2\sqrt{\alpha}} + 2\sqrt{\alpha} \tag{2.5}$$

$$\varsigma\left(\frac{d}{D}\right) \le g_2^a(\alpha), \quad g_2^a(\alpha) = \pi\left(\frac{R}{\sigma_0} - 1\right) + 2\sqrt{\alpha} + 2(1+\alpha)\operatorname{arcctg}\sqrt{\alpha} \tag{2.6}$$

Here

$$\varsigma\left(\frac{d}{D}\right) = \frac{4P}{\pi\sigma_0 D^2}\left(\frac{D}{d}\right)^{3/2} + \frac{2p}{\sigma_0}\left(\frac{d}{D}\right)^{1/2}, \quad \alpha = \frac{\Delta}{D}, \quad \beta = \frac{w_0}{HD\sigma_0} \tag{2.7}$$

where α is the reduced width of the cohesion zone and β is a constant of the composite material.

The minimum value of the function $g_1^a(\alpha)$ is $2\sqrt{\beta}$ and is reached when $\alpha = \alpha_0 = \beta/4$. Hence, and also from condition (2.5), we arrive at an inequality, which, in the model employed, defines one of the boundaries of the range of equilibrium cracks,

$$\varsigma\left(\frac{d}{D}\right) \ge 2\sqrt{\beta} \tag{2.8}$$

The second boundary is found from criterion (1.3) or the equivalent condition (2.6). We will assume that a certain value of $\alpha = \alpha_c$ corresponds to the critical state of the crack. In this case, the equality sign applies in condition (2.6). It then follows from conditions (2.5) and (2.6) that the quantity α_c is the root of the equation

$$g_1^a(\alpha) = g_2^a(\alpha) \tag{2.9}$$

Taking (2.8) into account we arrive at a double inequality, which defines the range of equilibrium states of the crack

$$2\sqrt{\beta} \le \varsigma\left(\frac{d}{D}\right) \le g_1^a(\alpha_c) \tag{2.10}$$

The reduced size of the cohesion zone α_c, for which the crack begins to grow, lies in the range $d_f/D \le \alpha_c \le \alpha_0$, where d_f is the fibre diameter. If $\alpha_c < d_f/D$, the crack begins to propagate only after all the bonds are broken. On the other hand, the inequality $\alpha_c > \alpha_0$ indicates that the lower limit of the range of equilibrium cracks is not described by this model.

We will denote the lower and upper boundaries of the diameters of the equilibrium cracks by d_l and d_u respectively. Inequality (2.9) can then be reduced to the following

$$d_l \le d \le d_u \tag{2.11}$$

In two special cases the quantities d_l and d_u are the simplest. For a crack which is opened solely by infinitely distant forces $\sigma_{33}^{\infty} = p$, it follows from formulae (2.7) and (2.10) that

$$d_l = D\frac{\beta\sigma_0^2}{p^2}, \quad d_u = D\left(\frac{\sigma_0 g_1^a(\alpha_c)}{2p}\right)^2 \tag{2.12}$$

When only concentrated forces P act at the centre of the crack, we obtain from formulae (2.7) and (2.10)

$$d_l = D\left(\frac{4P}{\pi\sigma_0 D^2 g_1^a(\alpha_c)}\right)^{2/3}, \quad d_u = D\left(\frac{2P}{\pi\sigma_0 D^2\sqrt{\beta}}\right)^{2/3} \tag{2.13}$$

Note that formulae (2.11)–(2.13) were derived for fixed values of the load parameters p and P. Alternative calculations for a fixed value of the crack diameter give load ranges which ensure the equilibrium state of the crack. The lower boundaries $p_l(P_l)$ and upper boundaries $p_u(P_u)$ of these intervals can be found from formulae (2.12) and (2.13) by replacing d_l or d_u by d, and $p(P)$ by $p_l(P_l)$ or $p_u(P_u)$ respectively.

The following possible scenario for the growth of a crack in the composite considered is obtained from the above results. If, for a fixed distant load p, the crack diameter d reaches value of d_u, while the dimensions of the bridged part Δ reaches value of $\Delta_c = \alpha_c D$, the fracture process transfers to the concluding stage of catastrophic fracture. Before this instant, a stable growth of the crack may occur with partially broken fibres, if condition (1.3) is violated and Eq. (1.1) is not satisfied. When equality (1.1) is satisfied, rupture of the outermost fibres begins in the cohesion zone. If the size of the cohesion zone falls in the range $\Delta_c \le \Delta \le \Delta_0$ ($\Delta_0 = \alpha_0 D$) and inequality (2.11) is satisfied, then, for a fixed value of the forces p the crack will be in an equilibrium state.

When concentrated forces P act at the centre of the crack, the nature of the growth of a bridged crack is the same as of a crack without bonds for a similar load,[29] namely, the growth of the crack is always stable. As follows from the second equality of (2.13), to sustain the fracture process a continuous increase in the value of P is necessary.

3. Comparison with plane strain

It follows from the previous discussion that the size of the bonded part of the bridged crack in the limit state $\Delta_c = \alpha_c D$ is exactly the same as when the action is due to forces at infinity p, concentrated forces P on the crack, and also a combination of these. It was shown in Ref. 2, that, in general, the value of Δ_c depends on the crack size, but, when the condition $\Delta \ll a$ is satisfied, it can be assumed to be constant. Hence, for this type of problem Δ_c is a constant of the material which defines the onset of the crack growth.

On the basis of the results obtained earlier in Ref. 13, we arrive at the same conclusion as regards a bridged crack, under plane strain. In this case, α_c has the same value as in the axisymmetric problem. In fact, for plane strain, α_c

satisfies the following equation, which is equivalent to Eq. (2.9)

$$g_1^p(\alpha) = g_2^p(\alpha);\quad g_j^p(\alpha) = \frac{2}{\pi}g_j^a(\alpha),\quad j = 1, 2 \tag{3.1}$$

In this case, relations (2.5)–(2.13) also hold in the case of plane strain for a crack of length $L = 2l$, if we replace g_j^a by g_j^p and d by L in them, with the exception of the second formulae of (2.5) and (2.6), and we replace β by $4\beta/\pi^2$ in formulae (2.8), (2.10), (2.12) and (2.13). It is also obvious that $d_l/d_u = L_l/L_u$. Here L_l is the lower boundary of the interval of the lengths of equilibrium cracks under plane strain and L_u is the upper boundary. We mean by P in the plane problem the sum of the uniformly distributed normal forces acting along the middle line of the crack surface on the part of length L. Note also that the quantities with the superscript a relate to the axisymmetric problem, while those with the superscript p relate to plane strain.

4. Fracture toughness

We will establish a relation between the fracture toughness of a composite, i.e. the limit values of the stress intensity factors ignoring the cohesion zone, and the critical size of this zone Δ_c in the axisymmetric problem and in the case of plane strain. We will write the corresponding asymptotic representations of the solutions for bridged cracks along the ρ lines as $\rho \to a$ and x_1 lines as $x_1 \to l$ in the form

$$\sigma_{33} = \frac{K^a}{\sqrt{2\pi(\rho - a)}} + O(1),\quad \sigma_{22} = \frac{K^p}{\sqrt{2\pi(x_1 - l)}} + O(1) \tag{4.1}$$

where l is the half-length of the crack in the plane problem while the factors K_a and K_p are found from the solutions of the boundary-value problems and are given by the formulae

$$K^a = \lim_{\rho \to a}\sigma_{33}(\rho, \theta)\sqrt{2\pi(\rho - a)},\quad K^p = \lim_{x_1 \to l}\sigma_{22}(x_1, 0)\sqrt{2\pi(x_1 - 1)}$$

Substituting the right-hand side of Eq. (2.2) in the first of the last relations, and the solution of the plane problem[13] into the second (taking into account the concentrated forces acting at the centre of the bridged crack), we obtain

$$\begin{aligned} K^a &= K_1^a - 2\sigma_0\sqrt{\frac{a^2 - b^2}{\pi a}},\quad K^p = K_1^p - \frac{2\sigma_0\sqrt{l}}{\sqrt{\pi}}\arccos\left(1 - \frac{\Delta}{l}\right) \\ K_1^a &= \frac{2p\sqrt{a}}{\sqrt{\pi}} + \frac{P}{\pi a\sqrt{\pi a}},\quad K_1^p = p\sqrt{\pi l} + \frac{P}{\sqrt{\pi l}} \end{aligned} \tag{4.2}$$

where K_1^a, K_1^p are the stress intensity factors, which define the asymptotic form of the stress-strain state at the edge of the corresponding crack when there are no end bonds.

In order to obtain the limit values of the factors K_{1c}^a and K_{1c}^p for cracks in a composite, we substitute the first expression from (4.1) into criterion (1.5), and the second into the similar condition (16) given in Ref. 13. Then, taking (4.2) and the conditions $\Delta \ll a$, $\Delta \ll l$ into account, we arrive at the following equations

$$K_{1c}^a = K_{1c}^p \equiv K_{1c} = K_0 + K_c;\quad K_0 = \frac{\pi E_3 R_m}{E_m}\sqrt{\frac{D}{2\pi}},\quad K_c = 4\sigma_0\sqrt{\frac{\Delta_c}{2\pi}} \tag{4.3}$$

Thus, the fracture toughness of a fibre composite is the sum of two terms: the quantities K_0, defined by the size of the fracture zone D at the edge of the crack, and the quantities K_c, defined by the critical size of the bonded part of the crack Δ_c. It should be noted that a similar formula, obtained previously in Ref. 2, for the fracture toughness of a composite, has a more complex structure and does not contain the dependence on the parameter Δ_c in explicit form; the quantity D also does not occur in it since the force fracture criterion was used instead of Novozhilov's criterion.

In addition to the presence of a cohesion zone, the effect of the fibres on the behaviour of the crack in a composite also manifests itself in the inequalities $E_3 \neq E_m$, $D \neq D_m$. According to relations (2.12), (2.13) and (4.3) even when

$\Delta_c = 0$ the presence of fibres affects both the boundaries of the range of equilibrium cracks d_l and d_u and the fracture toughness K_{1c}.

Of all the parameters that occur in the previous relations, the size of the fracture zone D remains unknown. Formula (4.3) can be used to calculate it, since the critical size of the cohesion zone Δ_c can be measured,[2] and the fracture toughness K_{1c} can be found in the standard way. Nevertheless, as follows from an analysis, presented at the end of this paper, for the ceramics considered below in a first approximation one can put $D = D_m$. Here the quantity D_m is found from the last formula of (1.5).

5. Example

We will apply the fracture model considered above to the materials SiC/SiC and SiC/CAS, which are slightly anisotropic ceramic composites with fibres of silicon carbide (SiC) with either the same matrix or a matrix of calcium-alumosilicate ceramics (CAS). The characteristics of the materials obtained using the data in Ref. 6 are shown in Table 1.

The ratio of Young's modulus in the isotropy plane to Young's modulus in a perpendicular direction is equal to $E_1/E_3 = 0.95$ for both materials. The strain law $\sigma(w)$ of the fibres[6] in the notation used here has the form

$$\frac{\sigma}{\sigma_A} = \frac{w}{d_f} \quad \text{when} \quad w \le w_B; \quad \frac{\sigma}{\sigma_A} = \sqrt{\frac{w - w_B}{d_f} + \left(\frac{w_B}{d_f}\right)^2} \quad \text{when} \quad w \ge w_B \tag{5.1}$$

where

$$\sigma_A = \frac{R_f}{d_f\sqrt{(w_0 - w_B)d_f + w_B^2}}$$

In Table 2 we show some characteristics of the equilibrium and limit states of a circular crack as a function of the size of the fracture zone D, calculated using formulae (5.1) and Table 1.

Table 1

Material	SiC/SiC	SiC/CAS
Young's modulus of the matrix, E_m (GPa)	300	100
Young's modulus of the fibre, E_f (GPa)	200	200
Young's modulus of the composite in the direction of the fibres, E_3 (GPa)	260	140
Poisson's ratio of the matrix, v_m	0.2	0.2
Poisson's ratio of the fibre, v_f	0.2	0.2
Tensile strength of the matrix, R_m (GPa)	2.83	0.96
Tensile strength of the fibre, R_f (GPa)	2	2
Fracture toughness of the matrix, K_{1c}^m (MPa m$^{1/2}$)	1.77	1.64
Force in the cohesion zone, σ_0 (GPa)	0.53	0.53
Volume fraction of the fibres, c	0.4	0.4
Opening of the crack at the boundary of the cohesion zone, $2w_0$ (μm)	2.06	4.08
Fibre diameter, d_f (μm)	7	1
Size of the fracture zone of the matrix when there are no fibres, D_m (μm)	0.25	1.8

Table 2

Material	SiC/SiC			SiC/CAS		
D/D_m	0.1	1	10	0.1	1	10
Δ_0/d_f	58.9	58.9	58.9	63.0	63.0	63.0
Δ_c/d_f	56.3	51.2	37.8	60.0	53.9	37.6
d_l/d_u	0.999	0.995	0.952	0.999	0.994	0.936
K_{1c} (MPa m$^{1/2}$)	17.3	17.6	18.6	18.0	18.7	20.9
K_0/K_c	0.029	0.096	0.353	0.041	0.138	0.521

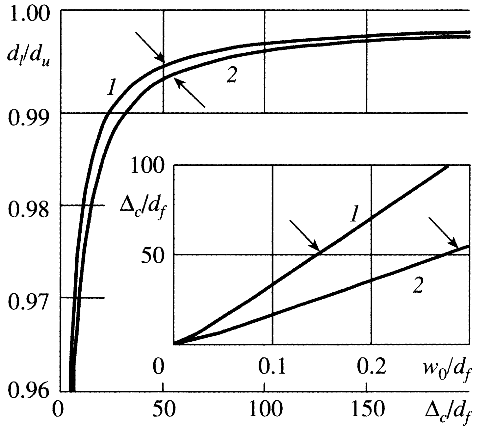

Fig. 4.

Allowing for the possibility of a variation in the limiting opening of a crack at the boundary of the cohesion zone, while keeping all the remaining characteristics of the composite unchanged, we arrive at relations of the critical size of the cohesion zone Δ_c as a function of this opening $2w_0$ and the ratio d_l/d_u as a function of Δ_c. These relations, which follow from relations (2.9) and (2.12), are shown in Fig. 4 for $D=D_m$ for SiC/SiC (curves 1) and SiC/CAS (curves 2). The arrows indicate points on the curve corresponding to the actual values of $2w_0$ (see Table 1).

It follows from Fig. 4 for Δ_c/d_f and Table 1 that for the ceramics considered $\Delta_c/d_m > \Delta_c/d_f \gg 1$. This means that the root of Eq. (2.9) lies in the region of large values, and we can obtain the following approximate relation for it

$$\Delta_c = \frac{D}{16}\left(\sqrt{A^2+\beta}-A\right)^2; \quad A = \pi\left(\frac{R}{\sigma_0}-1\right) \tag{5.2}$$

The relative error of this formula for values of Δ_c denoted by the arrows in Fig. 4 does not exceed 2%, and for the corresponding values of the ratio d_l/d_u it is 0.1%.

Using formulae (2.12) and (2.13) we constructed graphs, shown in Fig. 5, of the change in the reduced limiting loads p_u/R_f and P_u/P_f as a function of the diameter of a circular crack d (the continuous curves) and of the length of the crack L for plane strain (the dashed curves) for SeC/SiC ceramics for $D=D_m$. Here $P_f = \pi d_f^2 R_f/4$ is the tensile

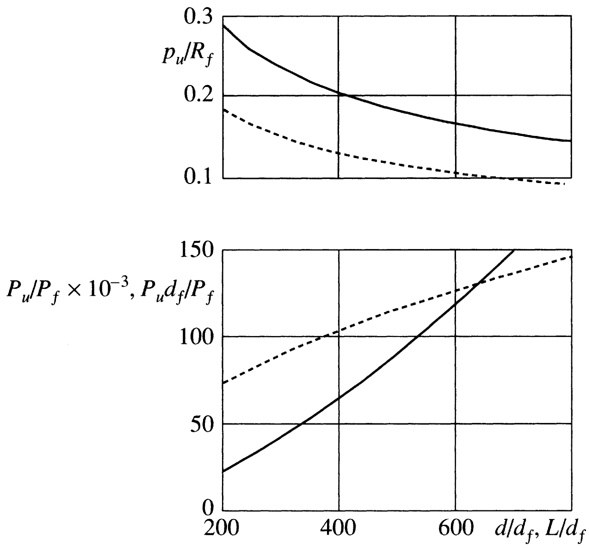

Fig. 5.

strength of the fibre, for which it ruptures. The similar curves for p_l/R_f and p_l/P_f practically coincide with these since $d_l/d_u \approx 1$ (see Table 2).

It is interesting to note that although certain characteristics of one composite differ considerably from the characteristics of the same type of the other composite (see Table 1), the limiting loads p_u and P_u for cracks in SiC/CAS ceramics are 3.3% higher in all than for cracks in the SiC/SiC ceramics. Because of this the curve shown in Fig. 5 can also apply to SiC/CAS ceramic, taking these differences in the limiting loads into account.

Discussion of the results of numerical experiments. According to Table 2 the size of the cohesion zone of a circular crack in the limit state when $D = D_m$ is approximately equal to $51d_f$ for SiC/SiC ceramics and $54d_f$ for SiC/CAS ceramics. Since, in the model used here, $\Delta/a \ll 1$, the crack size in the limit state must satisfy the condition $d_u \gg 104d_f$ ($d_u \gg 628\ \mu$m) for SiC/SiC and $d_u \gg 108d_f$ ($d_u \gg 754\ \mu$m) for SiC/CAS. Hence, in these materials the model constructed enables one to estimate the limit state of cracks with a diameter greater than 1 mm, which also explains the left limitation of the graphs in Fig. 5.

This limitation can be shifted to the left if we reduce the quantity β. In fact, in this case the root of Eq. (2.9) is reduced, and, by virtue of the monotonicity of the function $g_2(\alpha)$, the quantity $g_2(\alpha_c)$, which defines d_u, is reduced. The quantity β is directly proportional to the parameter w_0 and is inversely proportional to the parameters H, D and σ_0. The form of the curve of the critical size of the cohesion zone Δ_c against the quantity w_0 is shown in Fig. 4. Hence a way of reducing β is obvious. In order to reduce the value of w_0 we can increase, for example, the stiffness of the fibres or the fiber-matrix strength, adhesion and also reduce the strength of the fibre.

The problem of determining another important fracture characteristic, namely, the size of the fracture zone in front of the crack D, merits a separate discussion. The equality $D = D_m$ is justified, strictly speaking, for a crack in an isotropic body, the material of which is identical with the material of the matrix of the composite. For a fibre-reinforced material the value of D may be different, since the damage accumulation in the matrix occurs in the neighbourhood of the fibres, the presence of which influences the fracture process.

Hence, the quantity D is simultaneously a characteristic of the damage of the matrix and a characteristic of the structural non-uniformity of the composite, which manifests itself, in particular, in a non-uniformity of the stress-strain state in the neighbourhood of the fibre. The simplest way of considering this non-uniformity is to use a relation of the form $D = kD_m$.

The best basis for using the equality $D = D_m$ will be in the case of SiC/SiC ceramic. In fact, since the fracture of the matrix occurs between rows of fibres, the decisive factor for estimating the effect of the structure is the ratio D_m/u, where u is the distance between neighbouring rows of fibres. The greater this distance compared with the size of the fracture zone of an unreinforced matrix, the less should be the effect of the fibres on the fracture of the matrix in the composite. For hexagonal packing of the fibres, the value of u is found from the formula

$$u^2 = \pi d_f^2 \sqrt{3}/(8c)$$

Hence, it follows from Table 1 that $u = 9.13\ \mu$m for both types of ceramics, and $D_m/u = 0.027$ for SiC/SiC and $D_m/u = 0.2$ for Sic/CAS. Hence, for SiC/SiC we have $D_m \ll u$.

Nevertheless, it is necessary to point out that although the ratio D_m/u for SiC/CAS ceramics is an order of magnitude greater than for SiC/SiC, the dependence of the parameters on the size of the fracture zone D, presented in Table 2, is identical for both composites. The ratio d_l/d_u and the fracture toughness K_{1c} also react weakly to a considerable change in the value of D. The value of D has the greatest influence on the ratio K_0/K_c and on the critical size of the cohesion zone Δ_c. As D increases the value of Δ_c decreases, due to the increase in the difference $\Delta_0 - \Delta_c$.

Acknowledgements

We wish to thank the referees for their comments, which were taken into account in this version of the paper.

This research was supported financially by the Russian Foundation for Basic Research (05-01-00274, 03-01-00271).

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Translated by R.C.G.